Klausur 2

K 9)

a) leicht erraten: $x_1 = 1$, $x_2 = -1$, Division durch $(x-1)(x+1) = x^2-$

$$x^4 + x^3 - 3x^2 - x + 2 = (x^2-1)(x^2+x-2)$$

$$\begin{array}{l} x^4 + x^3 - 3x^2 - x + 2 \\ \underline{x^4 \mp x^2} \\ \quad x^3 - 2x^2 - x + 2 \\ \quad \underline{x^3 \quad\quad - x} \\ \quad\quad -2x^2 + 2 \\ \quad\quad \underline{-2x^2 + 2} \\ \quad\quad\quad 0 \end{array}$$

$$x^2 + x - 2 \overset{?}{=} 0 \iff \left(x^2 + x + \frac{1}{4}\right) - \frac{1}{4} - 2 = 0$$

$$\iff \left(x + \frac{1}{2}\right)^2 = \frac{9}{4} = \left(\frac{3}{2}\right)^2$$

$$\iff x + \frac{1}{2} = \pm\frac{3}{2}, \quad x_3 = 1, x_4 = -2$$

$$g(x) = x^4 + x^3 - 3x^2 - x + 2 = (x-1)^2(x+1)(x+2)$$

b) $h(x) = \dfrac{5x-2}{(x-1)(x+1)(x+2)} = \dfrac{a}{x-1} + \dfrac{b}{x+1} + \dfrac{c}{x+2} \quad \Big| \cdot (x$

$$\frac{5x-2}{(x+1)(x+2)} = a + b\,\frac{x-1}{x+1} + c\,\frac{x-1}{x+2} \quad \Big| \text{ einsetzen } x = 1$$

liefert: $\dfrac{3}{2 \cdot 3} = a \Rightarrow a = \dfrac{1}{2}$

analog mit $x+1$, $x=-1$; bzw $x+2$; $x=-2$

liefert: $b = \frac{7}{2}$, $c = -4$

$$\frac{5x-2}{(x-1)(x+1)(x+2)} = \frac{1}{2(x-1)} + \frac{7}{2(x+1)} - \frac{4}{x+2}$$

K10)

a) $\cosh(x) = \frac{1}{2}(e^{x}+e^{-x}) = \frac{1}{2}\sum_{n=0}^{\infty}\frac{x^n}{n!} + \sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!} = \frac{1}{2}\sum_{n=0}^{\infty}\frac{(1+(-1)^n)x^n}{n!}$

$= \sum_{k=0}^{\infty}\frac{x^{2k}}{(2k)!}$

$1+(-1)^n = \begin{cases} n \text{ gerade: } 2 \\ n \text{ ungerade: } 0 \end{cases}$

setze $n = 2k$, $k \in \mathbb{N}_0$

b) $f(x) = \sum_{n=0}^{\infty}\frac{2-(-1)^n}{3^n}\cdot x^n = 2\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n - \sum_{n=0}^{\infty}\left(-\frac{x}{3}\right)^n = 2\cdot\frac{1}{1-\frac{x}{3}} - \frac{1}{1+\frac{x}{3}}$

geometrische Reihen, Konvergenz für $\frac{|x|}{3} < 1$,

also für $|x| < 3$

$= \frac{6}{3-x} - \frac{3}{3+x} = \frac{6(3+x)-3(3-x)}{9-x^2}$

$= \frac{9+9x}{9-x^2}$

K11)

a) $N(t) = N_0\, e^{-\alpha t}$, $\alpha > 0$ $\quad N_0 = 10.000$

$2000 = N(5) = 10.000\cdot e^{-5\alpha}$, $e^{-5\alpha} = \frac{1}{5}$; $N(20) = 10.000\, e^{-20\alpha} = 10.000\left(e^{-5\alpha}\right)^4$

$= \frac{10000}{625} = \frac{100\cdot 100}{25\cdot 25} = 4\cdot 4 = 16$ //

b) $N(t) = N_0 e^{\alpha t}$, $\alpha > 0$ $\quad N(1) = 40$, $N(3) = 1000$

$40 = N_0\cdot e^{\alpha} \Rightarrow e^{\alpha} = \frac{40}{N_0}$ ⊛

$1000 = N_0\cdot e^{3\alpha} = N_0\cdot(e^{\alpha})^3 \overset{⊛}{=} N_0\cdot\frac{40^3}{N_0^3} = \frac{40^3}{N_0^2}$

$N_0^2 = \frac{40^3\cdot 40\cdot 40}{1000} = 4^3 = 64 \qquad N_0 = 8$

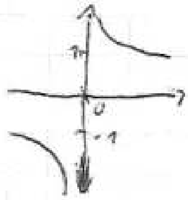
$00 = N(t) = 8e^{\alpha t}$, $e^{\alpha t} = 25$

$\Rightarrow t = \frac{\ln 25}{\alpha} = \frac{2 \cdot \ln 5}{\alpha} \underset{\uparrow}{=} \frac{2 \ln 5}{\ln 5} = 2$ (Tage)

④ $e^{\alpha} = 5$, also $\alpha = \ln 5$

12)

$\coth(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}}$



$\coth(-x) = \frac{e^{-x} + e^{x}}{e^{-x} - e^{x}} = -\frac{e^x + e^{-x}}{e^x - e^{-x}} = -\coth(x)$

Fall 1:

$0 < x < y$:

$\coth(x) > \coth(y) \Leftrightarrow \frac{e^x + e^{-x}}{e^x - e^{-x}} > \frac{e^y + e^{-y}}{e^y - e^{-y}} \Leftrightarrow (e^x + e^{-x})(e^y - e^{-y}) > (e^x - e^{-x})(e^y + e^{-y})$

$\Leftrightarrow \cancel{e^x e^y} + e^{-x}e^y - e^x e^{-y} - \cancel{e^{-x}e^{-y}} > \cancel{e^x e^y} - e^{-x}e^y + e^x e^{-y} - \cancel{e^{-x}e^{-y}}$

$\Leftrightarrow \cancel{2}e^{-x}e^y > \cancel{2}e^x e^{-y} \Leftrightarrow e^{-x+y} > e^{x-y}$

$\Leftrightarrow \frac{e^y}{e^x} > \frac{e^x}{e^y} \underset{\text{beide Nenner positiv}}{\Leftrightarrow} e^{2y} > e^{2x} \Leftrightarrow e^{2x} \Leftrightarrow \cancel{2}y > \cancel{2}x$ ✓

Fall 2:

$x < y < 0$:

Setze $x = -x'$, $y = -y'$, $x' > 0$, $y' > 0$

$-x' < -y' < 0 \Leftrightarrow x' > y' > 0$; somit $\coth(x') < \coth(y')$, also

$\coth(x) = \coth(-x) < \coth(-y) = -\coth(y)$

$\Rightarrow \coth(x) > \coth(y)$

)

$\lim_{x \to -\infty} \frac{e^x - 6x^3 - x + 2}{2x^2 + 9x^3} = \lim_{x \to -\infty} \frac{\frac{\exp(x)}{x^3} - 6 - \frac{1}{x^2} + \frac{2}{x^3}}{\frac{2}{x} + 9} \underset{\uparrow}{=} -\frac{6}{9} = -\frac{2}{3}$

b) $\lim\limits_{x\to\infty} \left(x^2-\sqrt{x^4-x^2+1}\right) = \lim\limits_{x\to\infty} \frac{\cancel{x^4}-\cancel{x^4}+x^2-1}{x^2+\sqrt{x^4-x^2+1}}$

$= \lim\limits_{x\to\infty} \frac{1-\frac{1}{x^2}}{1+\sqrt{1-\frac{1}{x^2}+\frac{1}{x^4}}} = \frac{1}{1+1} = \frac{1}{2}$

c) $\lim\limits_{x\to\infty} \sqrt{x}^{\frac{1}{x}} = \lim\limits_{x\to\infty} \exp\left(\frac{1}{x}\ln\sqrt{x}\right) = \exp \underbrace{\lim\limits_{y\to\infty}\left(\frac{\ln y}{y^2}\right)}_{0} = \exp(0) = 1$

exp stetig
auf $\mathbb{R}$
setze $y := \sqrt{x}$

d) $\lim\limits_{x\to 0(+)} \sqrt{x}^{\frac{1}{x}} = \lim\limits_{x\to 0(+)} \exp\left(\frac{1}{x}\ln\sqrt{x}\right) = \exp \lim\limits_{y\to 0(+)}\left(\frac{\ln y}{y^2}\right) = \exp(-\infty) = 0$

e) $\lim\limits_{x\to 0(+)} \left(1-3\cdot\sqrt{x}\right)^{\frac{1}{\sqrt{x}}} \underset{y:=\frac{1}{\sqrt{x}}}{=} \lim\limits_{y\to\infty}\left(1-\frac{3}{y}\right)^y = e^{-3} = \frac{1}{e^3}$

Vorlesung

f) $\lim\limits_{x\to 0} \frac{\cos(x)-1+\frac{x^2}{2}}{x^4} = \lim\limits_{x\to 0} \frac{\frac{x^4}{4!}-\frac{x^6}{6!}\pm\dots}{x^4} = \lim\limits_{x\to 0} \frac{1}{4!}-\underbrace{\frac{x^2}{6!}\pm\dots}_{\text{mit Potenzen von } x}$

$= \frac{1}{24}$

K14)

a) $z = -3-4i = |z|e^{i\arg(z)} = 5e^{i\arg(z)}, \arg(z) = 2\pi - \underbrace{\arccos\frac{-3}{5}}_{\text{Taschenrechner}}$

$= 2\pi - 2{,}21$

b) $z_1 = e^{-i\cdot\frac{3\pi}{2}} = \cos\left(-\frac{3\pi}{2}\right) + i\sin\left(-\frac{3\pi}{2}\right) = \cos\frac{3\pi}{2} - i\sin\frac{3\pi}{2} = i$

$z_2 = 2\cdot e^{i\frac{\pi}{2}} = 2(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}) = 2i$

$z_3 = 3\cdot e^{-i\frac{\pi}{4}} = 3(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}) = 3(\frac{1}{2}\sqrt{2} - i\frac{1}{2}\sqrt{2}) = \frac{3}{2}\sqrt{2}$

$z_1^4 = e^{-i\frac{3\pi}{2}\cdot 4} = e^{-6\pi i} = \underbrace{\cos 6\pi}_{1} - \underbrace{i\sin 6\pi}_{0} = 1$

$z_1\cdot z_2 = 2e^{i(\frac{\pi}{2}-\frac{3\pi}{2})} = 2e^{-i\pi} = 2(\cos\pi - i\sin\pi) = -2$

$z_3^2 = 9\cdot e^{-i\frac{\pi}{2}} = 9(\cos\frac{\pi}{2} \cdot i\sin\frac{\pi}{2}) = -9i$